Symmetry Breaking Hopf Bifurcations in Equations with O(2) Symmetry with Application to the Kuramoto–Sivashinsky Equation

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In problems with $O(2)$ symmetry, the Jacobian matrix at nontrivial steady state solutions with D_n symmetry always has a zero eigenvalue due to the group orbit of solutions. We consider bifurcations which occur when complex eigenvalues also cross the imaginary axis and develop a numerical method which involves the addition of a new variable, namely the velocity of solutions drifting round the group orbit, and another equation, which has the form of a phase condition for isolating one solution on the group orbit. The bifurcating branch has a particular type of spatio-temporal symmetry which can be broken in a further bifurcation which gives rise to modulated travelling wave solutions which drift around the group orbit. Multiple Hopf bifurcations are also considered. The methods derived are applied to the Kuramoto–Sivashinsky equation and we give results at two different bifurcations, one of which is a multiple Hopf bifurcation. Our results give insight into the numerical results of Hyman, Nicolaenko, and Zaleski (*Physica D* **23,** 265, 1986).   © 1997 Academic Press

1. INTRODUCTION

We consider Hopf bifurcations which occur on nontrivial steady state solutions which have D_n symmetry in systems with $O(2)$ symmetry. The usual equivariant Hopf theorem [12] cannot be applied in this situation since there is a zero eigenvalue of the Jacobian at every solution due to the group orbit of solutions.

Aston, Spence, and Wu [6] considered a similar type of problem but associated with an eigenvalue passing through zero and showed that a bifurcating branch of travelling wave (TW) solutions occurs which is associated with a drift around the group orbit. The approach used was to modify the system by adding a phase condition to eliminate the degenerate zero eigenvalue so that standard theory could be applied. They also considered the situation when a Hopf bifurcation occurs on a branch of TW solutions and showed that it gives rise to modulated travelling wave (MTW) solutions [7].

Krupa [21] considers the related but more general problem of bifurcation from group orbits for problems which are equivariant with respect to subgroups of $O(n)$. In this case, the degeneracy is dealt with by splitting the vector field into two parts, one tangent to the group orbit and the other normal to it. A standard bifurcation analysis is then performed on the normal vector field and the results are interpreted for the whole vector field. He considers the Kuramoto–Sivashinsky equation with $O(2)$ symmetry as an example but does not present any new numerical results.

Landsberg and Knobloch [22] also considered this problem and showed that at a Hopf bifurcation on the steady state solution, so-called direction reversing travelling wave (DRTW) solutions bifurcate. They transformed the problem into canonical coordinates [8] so that one equation, associated with the drift around the group orbit, decouples from the others thus separating out the degeneracy. Proctor and Weiss [24] and Matthews, Proctor, Rucklidge, and Weiss [25] have also considered this problem in the context of nonlinear magnetoconvection. They refer to the bifurcating periodic solutions as pulsating waves. They also mention that further bifurcation yields MTW solutions. However, they did not do any analysis of the bifurcations involved.

The methods using canonical coordinates for this problem have been considered in more detail by Amdjadi [1]. However, the drawback with these methods is that they cannot be usefully applied directly to partial differential equations. Thus, the approach we use in this paper is similar to that of Aston, Spence, and Wu [6, 7], namely the addition of a phase condition and an extra variable to eliminate the degeneracy due to the group orbit. The Hopf bifurcation under consideration is then a standard symmetry-breaking Hopf bifurcation giving rise to time periodic solutions with a particular spatio-temporal symmetry. If this symmetry is broken in a further bifurcation, modulated travelling wave solutions arise. This method is then applied to the Kuramoto–Sivashinsky equation and numerical results are obtained in two different parameter regimes. This enables some of the numerical results of Hyman, Nicolaenko, and Zaleski [13] to be interpreted in a precise way.

2. BIFURCATIONS

Consider the problem

$$\dot{u} = g(u, \lambda), \quad g: X \times \mathbf{R} \to X, \qquad (2.1)$$

where X is a Hilbert space. We suppose that this problem has $O(2)$ symmetry so that

$$g(\gamma u, \lambda) = \gamma g(u, \lambda) \quad \text{for all } \gamma \in O(2),$$

where the group $O(2)$ is generated by the rotations r_α, $\alpha \in [0, 2\pi)$ and the reflection s. Assuming time periodic boundary conditions, the full symmetry of the problem is then $O(2) \times S^1$, where the additional S^1 symmetry is due to time translations.

We decompose the space X as $X = X^s \oplus X^a$, where X^s and X^a are the symmetric and antisymmetric spaces with respect to the reflection s respectively. In problems with $O(2)$ symmetry, there are typically many branches of symmetric steady state solutions contained in X^s. However, due to the continuous spatial symmetry, there are group orbits of nontrivial steady states (or relative equilibria) passing through each symmetric steady state. In this case, the linearisation $g_u(u, \lambda)$ evaluated at a nontrivial steady state has a nontrivial null space for all values of λ where the nullvector is the tangent vector to the group orbit [6]. This degeneracy due to the group obit means that standard bifurcation analysis cannot be used to study bifurcations in which either a real eigenvalue passes through zero or a pair of complex conjugate eigenvalues cross the imaginary axis.

The reason for using canonical coordinates as described in [1, 22] is to change variables in order to factor out the degeneracy due to the group orbit, giving a reduced system of nondegenerate equations, together with an additional equation which decouples from the rest. This method works well for small systems of ordinary differential equations but is of no practical use for partial differential equations. An alternative way of removing the degeneracy is to add an additional equation, which fixes one solution on the group orbit and to include an additional variable, namely the drift velocity around the group orbit. This method is amenable to an analysis of the bifurcations involved using standard techniques, since the degeneracy has been removed, and also leads to numerical methods for dealing with such bifurcations.

We seek solutions of the form

$$u(t) = r_{ct}x(t) \qquad (2.2)$$

which allows for time periodic solutions $x(t)$ with a drift around the group orbit superimposed on them, with velocity c. Substituting (2.2) into (2.1) gives

$$\dot{x} = g(x, \lambda) - cAx = \tilde{g}(x, c, \lambda), \qquad (2.3)$$

where Ax is the tangent vector to the group orbit defined by

$$Ax = \frac{d}{d\alpha}(r_\alpha x)|_{\alpha=0}.$$

By adding a phase condition of the form

$$\dot{c} = p(x, c, \lambda),$$

the zero eigenvalue due to the rotational symmetry can be eliminated and so standard theory can then be applied (see [7]). Since we want solutions with $c(t)$ constant in time, the right-hand side of the phase condition must be independent of both time t and c. Thus, we choose the phase condition to be

$$\dot{c} = p(\bar{x}, \lambda), \qquad (2.4)$$

where $\bar{x}$ is the time average of $x(t)$ over one period T defined by

$$\bar{x} = \frac{1}{T}\int_0^T x(t)\,dt.$$

The solution of the differential equation (2.4) with periodic boundary conditions is given by $c(t) = $ constant, as required, and so (2.4) becomes

$$p(\bar{x}, \lambda) = 0$$

which fixes the spatial phase of the solutions on the group orbit. Thus, we consider the system

$$\dot{y} = G(y, \lambda) = \begin{bmatrix} \tilde{g}(x, c, \lambda) \\ p(\bar{x}, \lambda) \end{bmatrix}, \qquad (2.5)$$

where $y = (x, c) \in Y = X \times \mathbf{R}$. One simple choice of the phase function p is

$$p(\bar{x}, \lambda) = \langle \ell, \bar{x} \rangle, \quad \ell \in X,$$

where $\langle\,,\,\rangle$ defines an inner product on X.

In order to fix the spatial phase the phase function p must satisfy the nondegeneracy condition

$$\langle p_{\bar{x}}(\bar{x}_0, \lambda_0), A\bar{x}_0 \rangle \neq 0, \qquad (2.6)$$

or equivalently,

$$\langle \ell, A\overline{x}_0 \rangle \neq 0 \qquad (2.7)$$

for the simple form of the phase function [7].

The eigenvalues of G_y are simply related to those of g_x. The following result is proved in [9].

THEOREM 2.1. *Suppose that (x_0, c_0, λ_0) is a solution of $G((x, c), \lambda) = 0$. If the eigenvalues of $\tilde{g}_x(x_0, c_0, \lambda_0)$ are σ_i, $i = 1, ..., n$ with $\sigma_n = 0$, then the eigenvalues of $G_y((x_0, c_0), \lambda_0)$ are σ_i, $i = 1, ..., n - 1$ and $\pm\delta$, where*

$$\delta = [- \langle p_{\overline{x}}(\overline{x}_0, \lambda_0), A\overline{x}_0 \rangle]^{1/2}.$$

Clearly, if $g_x(x_0, c_0, \lambda_0)$ has eigenvalues $\pm i\omega_0$ then so has $G_y((x_0, c_0), \lambda_0)$. Also, note that the nondegeneracy condition (2.6) is precisely $\delta \neq 0$.

For the simple choice of phase function, the additional eigenvalues of $G_y((x_0, c_0), \lambda_0)$ are $\pm\delta$, where $\delta = [- \langle \ell, Ax_0 \rangle]^{1/2}$. Thus, if $\langle \ell, Ax_0 \rangle$ is positive, then $\pm\delta$ will be on the imaginary axis. Numerically, this will cause confusion with the eigenvalues which cross the imaginary axis at the Hopf bifurcation point. Thus it is important to choose ℓ so that $\langle \ell, Ax_0 \rangle$ is negative to ensure that the additional eigenvalues lie on the real axis. Detection of this type of bifurcation can be achieved by using AUTO [11] on the system $G(y, \lambda) = 0$. We note that on a steady state solution, x is independent of time. Thus $\overline{x} = x$ and the phase condition reduces to $\langle \ell, x \rangle = 0$ which is a simple algebraic equation that is easily implemented.

The reason for introducing the phase condition is to eliminate the continuous spatial symmetry. However, it is important that other symmetries are retained. It is easily verified that if the phase function p satisfies

$$p(\overline{sx}, \lambda) = -p(\overline{x}, \lambda),$$
$$p(\overline{r_{2\pi/n}x}, \lambda) = p(\overline{x}, \lambda), \quad n \in \mathbf{Z}^+, \qquad (2.8)$$

then $G(y, \lambda)$ is equivariant with respect to the reflection S, the discrete rotation $R_{2\pi/n}$, and the time translation $\theta \in S^1$ which act on $y = (x, c)$ by

$$S \begin{bmatrix} x \\ c \end{bmatrix} = \begin{bmatrix} sx \\ -c \end{bmatrix}, \quad R_{2\pi/n} \begin{bmatrix} x \\ c \end{bmatrix} = \begin{bmatrix} r_{2\pi/n}x \\ c \end{bmatrix},$$

$$\theta \begin{bmatrix} x(t) \\ c(t) \end{bmatrix} = \begin{bmatrix} x(t + T\theta/2\pi) \\ c(t + T\theta/2\pi) \end{bmatrix}.$$

Thus, the symmetry of G is $D_n \times S^1$ for an appropriate choice of p, where D_n is the dihedral group generated by the rotations $R_{2\pi/n}$ and the reflection S. The theory related to Hopf bifurcations in this case is well developed [12]. Of particular interest are Hopf bifurcations which break the reflectional symmetry S. These can occur in two different

ways. First, if the Hopf bifurcation is associated with the one-dimensional irreducible representation $R = I, S = -I$, then the eigenvalues $\pm i\omega_0$ will generically be simple and there will be a single bifurcating branch of time-periodic solutions. Since both S and $\pi \in S^1$ act as $-I$ on the eigenspace, the bifurcating branch will have the spatial-temporal symmetry $(S, \pi) \in D_n \times S^1$. Note that $(S, \pi)^2 = (I, 0)$ and so this is a type of reflection operator. The interpretation of this symmetry is that a shift in time by half a period is equivalent to acting with S. We note from the definitions of S and θ that if y has (S, π) symmetry, then

$$-c (t + T\theta/2\pi) = c(t)$$

and since c must be a constant, this implies that $c = 0$. Thus, these solutions do not drift around the group orbit and therefore correspond to time-periodic solutions of (2.1). However, they do break out of the symmetric space associated with S and so the solutions oscillate either side of the reflection invariant space. Thus, we refer to these solutions as *oscillating waves*.

The second way in which such solutions arise is due to a Hopf bifurcation associated with a two-dimensional irreducible representation. In this case, there are three bifurcating branches of time-periodic solutions, each with different symmetries [12]. By considering the effect of the symmetries for each branch on the velocity component c, we show (see Section 3) that $c = 0$ on two of the branches while $c \neq 0$ on the third. Thus, two of the bifurcating branches consist of time-periodic solutions of which one is contained in the reflection invariant space while the other is a branch of oscillating waves. The third branch with $c \neq 0$ corresponds to modulated travelling waves. This is consistent with the results of Krupa [21] who considered this problem in a different way.

Numerically, once a Hopf bifurcation point has been detected, a starting solution on the branch of periodic solutions can be obtained for the variable x using the information contained in the eigenfunctions. We note, however, that the eigenfunctions of the linearisation of $G(y, \lambda) = 0$ which includes the simple phase condition $\langle \ell, x \rangle = 0$ are not appropriate for constructing the initial solution. This is due to the fact that the linear operator $G_y((x_0, 0), \lambda_0)$ involves the time averaging term. We now address this issue.

If $x_0 \in X^s$, then we denote the restriction of $\tilde{g}_x(x_0, 0, \lambda_0)$ to X^s and X^a by $\tilde{g}_x^s(x_0, 0, \lambda_0)$ and $\tilde{g}_x^a(x_0, 0, \lambda_0)$, respectively [30]. Then

$$G_y((x_0, 0), \lambda_0) = \begin{pmatrix} \tilde{g}_x^s & 0 & 0 \\ 0 & \tilde{g}_x^a & -Ax_0 \\ 0 & \langle \ell^a, \dashv \rangle & 0 \end{pmatrix},$$

where $Ax_0 \in X^a$ and $\ell = \ell^a \in X^a$ to ensure that G has the symmetry S. Note that the reflectional symmetry S can also be used to decompose $Y = Y^s \oplus Y^a$. Hence, the linearization of (2.5) with respect to y evaluated at a steady state solution has the form

$$\dot{\Phi} = G_y(y_0, \lambda_0)\Phi, \qquad (2.9)$$

where $\Phi = [\Phi^s, \Phi^a, \Phi^c] \in X^s \times X^a \times \mathbf{R}$ and $y_0 = (x_0, 0)$.

THEOREM 2.2. (i) *If g_x^s has eigenalues $\pm i\omega_0$ then the solution of (2.9) is $\Phi(t) = [\Phi_s(t), 0, 0] \in Y^s$, where $\dot{\Phi}_s = g_x^s(x_0, \lambda_0)\Phi_s$.*

(ii) *If g_x^a has eigenvalues $\pm i\omega_0$ then the solution of (2.9) is $\Phi(t) = [0, \Phi_a(t), 0] \in Y^a$, where $\Phi_a(t)$ satisfies $\dot{\Phi}_a = g_x^a(x_0, \lambda_0)\Phi_a$ and is constructed using the eigenfunctions associated with the eigenvalues $\pm i\omega_0$.*

Proof. (i) This is a standard result since $g_x^s(x_0, \lambda_0)$ has only got the eigenvalues $\pm i\omega_0$ on the imaginary axis and $\langle \ell^a, \overline{\Phi}_s \rangle = 0$, which we include for later reference.

(ii) Expanding (2.9), we obtain

$$\dot{\Phi}^s = g_x^s(x_0, \lambda_0)\Phi^s,$$
$$\dot{\Phi}^a = g_x^a(x_0, \lambda_0)\Phi^a - \Phi^c Ax_0,$$
$$\dot{\Phi}^c = \langle \ell^a, \overline{\Phi}^a \rangle.$$

The first equation implies that $\Phi^s = 0$, since g_x^s has no eigenvalues on the imaginary axis. Since $\overline{\Phi}^a$ is independent of time, then $\langle \ell^a, \overline{\Phi}^a \rangle = k_1$, where k_1 is constant. Thus $\dot{\Phi}^c = k_1$, giving $\Phi^c = k_1 t + k_2$ for some constant k_2. The periodic boundary conditions imply that $k_1 = 0$, since $k_1 t$ is only time periodic if $k_1 = 0$, and so $\Phi^c = k_2$. Therefore, the second equation is now

$$\dot{\Phi}^a = g_x^a(x_0, \lambda_0)\Phi^a - k_2 Ax_0. \qquad (2.10)$$

It is easily verified that $g_x^a(x_0, \lambda_0)Ax_0 = 0$ and so the solution of the homogeneous equation $\dot{\Phi}^a = g_x^a(x_0, \lambda_0)\Phi^a$ is $\Phi^a(t) = \Phi_a(t) + \beta Ax_0$ for any $\beta \in \mathbf{R}$ and where $\Phi_a(t)$ was defined in the statement of the theorem. To find the particular solution of the above system we assume that $\Phi^a(t) = v$ is a solution, where v is a constant function. Then

$$g_x^a(x_0, \lambda_0)v - k_2 Ax_0 = 0.$$

If ψ_0 is the left eigenvector of $g_x^a(x_0, \lambda_0)$ corresponding to the zero eigenvalue, then

$$-k_2 \langle \psi_0, Ax_0 \rangle = 0.$$

This implies that $k_2 = 0$, since generically $\langle \psi_0, Ax_0 \rangle \neq 0$ and so $\Phi^c(t) = 0$, as anticipated, since $c = 0$ on the whole

bifurcating branch. Thus (2.10) is now a homogeneous equation with solution $\Phi^a(t) = \Phi_a(t) + \beta Ax_0$. Now $\overline{\Phi}^a = \beta Ax_0$, since the time-dependent part of $\Phi^a(t)$ involves $\sin \omega_0 t$ and $\cos\omega_0 t$, both of which have zero mean. From $\langle \ell^a, \overline{\Phi}^a \rangle = k_1 = 0$ we have that $\beta \langle \ell^a, Ax_0 \rangle = 0$. Hence, the nondegeneracy condition (2.7) implies that $\beta = 0$. Therefore, $\Phi^a = \Phi_a$ as claimed. ∎

In the symmetry-breaking case, the solution on the bifurcating branch is given by

$$x(t) = x^s + \alpha\Phi_x(t) + O(\alpha^2),$$
$$c = 0, \qquad (2.11)$$
$$\lambda = \lambda_0 + O(\alpha^2),$$

where $\Phi_x(t) = [0, \Phi_a(t)]$. An initial approximation to the solution is obtained by truncating the higher order terms and choosing a small value for α.

To compute the periodic solutions we use the spatial phase condition $\langle \ell^a, \overline{x} \rangle = 0$, together with a standard temporal phase condition which is built into AUTO. The system is then solved for x and the scalar variables c and T. The possible further bifurcation occurs in this system with no modifications and so the standard AUTO procedure can be used for detection and swapping branches.

3. THE KURAMOTO–SIVASHINSKY EQUATION

The Kuramoto–Sivashinsky (KS) equation describes many physical phenomena including reaction–diffusion problems (Kuramoto [18, 19]), flame fronts in combustion problems (Sivashinsky [27, 28]), the dynamics of viscous-fluid films flowing along walls (Sivashinsky and Michelson [29], Shlang and Sivashinsky [26]), cross-roll and zigzag instabilities in convective patterns (Kuramoto [20]), as well as several other physical phenomena (see [17]). Much effort has gone into the study of the KS equation. In particular, many numerical results have been obtained by Hyman, Nicolaenko, and Zaleski [13] and our results give insight into some of their computations.

The equation we consider is given by

$$U_t + 4U_{xxxx} + \lambda(U_{xx} + \tfrac{1}{2} U_x^2) = 0,$$
$$U(0, t) = U(2\pi, t), \qquad (3.1)$$

where the spatial variable has been rescaled so that we can consider 2π-periodic solutions. The parameter λ is then related to the true period L.

We first transform this equation into a more convenient form, following [13]. The mean value of the solution to Eq. (3.1) is $m(t) = (1/2\pi) \int_0^{2\pi} U(x, t)\, dx$ and so

$$\frac{dm}{dt} = \frac{1}{2\pi} \int_0^{2\pi} U_t(x, t)\, dx.$$

Integrating (3.1) on the interval $[0, 2\pi]$ implies that

$$\frac{dm}{dt} = -\frac{\lambda}{4\pi} \int_0^{2\pi} U_x^2\, dx.$$

To normalise the drift to zero we define

$$u(x, t) = U(x, t) - m(t),$$

where u has zero mean. Therefore we obtain the equation

$$u_t + 4u_{xxxx} + \lambda\left(u_{xx} + \frac{1}{2} u_x^2\right) - \frac{\lambda}{4\pi} \int_0^{2\pi} u_x^2\, dx = 0. \quad (3.2)$$

To eliminate the integral term and to avoid further numerical difficulties which we describe in the Appendix, we define $v = u_x$ and differentiate (3.2) with respect to x to give

$$-v_t = g(v, \lambda) := 4v_{xxxx} + \lambda(v_{xx} + vv_x) = 0,$$
$$g : H_0^4(0, 2\pi) \to H_0^0(0, 2\pi), \quad (3.3)$$

where $H_0^m(0, 2\pi)$ is the space of 2π-periodic functions with zero mean whose derivatives up to and including the mth are square integrable. For ease of notation, we define $X = H_0^4(0, 2\pi)$. It is easily verified that this equation is equivariant with respect to the action of $O(2)$ defined by

$$r_\alpha v(x, t) = v(x + \alpha, t),$$
$$sv(x, t) = -v(-x, t).$$

It is well known that steady state bifurcations from the trivial solution occur at $\lambda = 4n^2$, $n \in \mathbf{Z}^+$, resulting in branches of solutions contained in $X^{D_n} \times \mathbf{R}$, where X^{D_n} is the fixed point space associated with the dihedral group D_n, generated by s and $r_{2\pi/n}$. We refer to the nth such branch as primary branch n.

Substituting $v = r_{\alpha(t)}\tilde{v}$ into (3.3) and dropping the tildes gives

$$v_t + 4v_{xxxx} + \lambda(v_{xx} + vv_x) + c(t)Av = 0, \quad (3.4)$$

where $c(t) = \dot{\alpha}(t)$ and the linear operator A is given by $Av = v_x$. We include the phase condition

$$\dot{c}(t) = \langle \ell, \overline{v} \rangle, \quad (3.5)$$

where $\overline{v}$ is the time average of v over one period. The periodic boundary conditions then imply that $\dot{c}(t) = 0$ (see Section 2). We write (3.4) and (3.5) as

$$y_t = G(y, \lambda), \quad (3.6)$$

where $y = (v, c) \in Y = X \times \mathbf{R}$ and define

$$S\begin{pmatrix} v \\ c \end{pmatrix} = \begin{pmatrix} sv \\ -c \end{pmatrix}, \quad R\begin{pmatrix} v \\ c \end{pmatrix} = \begin{pmatrix} r_{2\pi/n} v \\ c \end{pmatrix}.$$

Then $G(y, \lambda)$ is equivariant with respect to the dihedral group D_n generated by R and S if $r_{2\pi/n}\ell = \ell$ and $s\ell = -\ell$ (see (2.8)). We refer to solutions of (3.4) which satisfy $sv = v$ as symmetric solutions. Clearly any solution of $G(y, \lambda) = 0$ which satisfies $Sy = y$ must have $c = 0$ and thus involves a symmetric steady state solution of (3.4). If the solution also satisfies $Ry = y$ then the steady state solution has D_n symmetry.

We now consider the possibility of time periodic branches of solutions bifurcating from a branch of nontrivial steady state solutions. We assume that along a primary branch of solutions of (3.6), there is a point (y_0, λ_0), where $y_0 = (v_0, 0)$, such that $G_y(y_0, \lambda_0)$ has eigenvalues $\pm i\omega_0$. This primary branch lies in the fixed point space $Y^{D_n} \times \mathbf{R}$ for some n.

Now the linearisation $G_y(y_0, \lambda_0)$ decomposes into diagonal blocks on the isotypic components of the space Y, which are associated with the irreducible representations of D_n [5]. The nontrivial irreducible representations of D_n are given by

(i) $R = I, S = -I$,

(ii) $R = -I, S = I$ (n even),

(iii) $R = -I, S = -I$ (n even),

(iv) $R = \begin{pmatrix} \cos(2\pi m/n) & \sin(2\pi m/n) \\ -\sin(2\pi m/n) & \cos(2\pi m/n) \end{pmatrix}$,

$$S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{matrix} m = 1, ..., \frac{1}{2}n - 1 & (n \text{ even}) \\ = 1, ..., \frac{1}{2}(n - 1) & (n \text{ odd}). \end{matrix}$$

There is also the trivial irreducible representation $R = I, S = I$. All of these representations are absolutely irreducible and the corresponding isotypic components of Y are

$$Y_0 = \left(\sum_{k=1}^{\infty} a_k \sin nkx, 0\right) = Y^{D_n},$$

$$Y_1 = \left(\sum_{k=0}^{\infty} a_k \cos nkx, c\right),$$

$$Y_2 = \left(\sum_{k=0}^{\infty} a_k \sin\left(nk + \frac{n}{2}\right)x, 0\right) \quad (n \text{ even})$$

$$Y_3 = \left(\sum_{k=0}^{\infty} a_k \cos\left(nk + \frac{n}{2}\right)x, 0 \right) \quad (n \text{ even})$$

$$Y_{4m} = \left(\sum_{k=0}^{\infty} [a_k \sin(nk+m)x + b_k \sin(n(k+1)-m)x], 0 \right)$$

$$\oplus \left(\sum_{k=0}^{\infty} [c_k \cos(nk+m)x + d_k \cos(n(k+1)-m)x], c \right),$$

$$m = \begin{cases} 1, ..., \frac{1}{2}n - 1 & (n \text{ even}) \\ 1, ..., \frac{1}{2}(n-1) & (n \text{ odd}). \end{cases}$$

Thus $Y = Y_0 \oplus Y_1 \oplus Y_2 \oplus Y_3 \oplus Y_{4m}$ (see [5]).

Linearising (3.6) about (y_0, λ_0) on the primary branch gives

$$\dot{\Phi} = G_y(y_0, \lambda_0)\Phi. \tag{3.7}$$

Note that $G_y(y_0, \lambda_0)$ is not independent of time due to the phase condition. If $G_y(y_0, \lambda_0)$ has eigenvalues $\pm i\omega_0$ then Eq. (3.7) has a solution of the form $\Phi(x, t) = [\Phi_1(x, t), 0]^{\mathrm{T}}$, where $\Phi_1(x, t) = e^{i\omega_0 t}\phi_1(x)$ is the solution of the linearisation of (3.4) associated with the imaginary eigenvalues (see Theorem 2.2). We assume that time is rescaled so that the imaginary eigenvalues become $\pm i$.

Now suppose that the eigenfunctions $\phi_1(x) = \phi_{1r}(x) \pm i\phi_{1j}(x)$ satisfy $\phi_{1r}(x), \phi_{1j}(x) \in Y_1$. Thus the corresponding real eigenspace is $E_i = \mathrm{sp}\{\phi_{1r}, \phi_{1j}\} \subset Y_1$ and is two dimensional. Now $\pi \in S^1$ acts as $-I$ on E_i and since $E_i \subset Y_1$ we also have that $S\phi_1 = -\phi_1$, $R\phi_1 = \phi_1$. Hence, $(S, \pi) = S_1 \in D_n \times S^1$ fixes $\phi_1 \in E_i$. The action $R_1 = (R, 0)$, where $R_1^n = (I, 0) \in D_n \times S^1$, also fixes ϕ_1. Note that $S_1^2 = (I, 0)$ and $S_1 R_1 = R_1^{-1} S_1$. Hence, the isotropy group Σ_1 generated by S_1 and R_1 is the symmetry group of the eigenspace E_i and is isomorphic to D_n. Since $\dim(Y^{\Sigma_1} \cap E_i) = 2$, by the equivariant Hopf theorem there exists a branch of periodic solutions bifurcating from the steady state branch having Σ_1 as its group of symmetries.

Similarly, if $\phi_{1r}, \phi_{1j} \in Y_2$ then $E_i \subset Y_2$ and is two-dimensional. The symmetry group Σ_2 of E_i is then generated by $S_2 = (S, 0)$ and $R_2 = (R, \pi)$ which is again isomorphic to D_n. Hence the equivariant Hopf theorem implies a branch of solutions with symmetry $\Sigma_2 \subset D_n \times S^1$.

Finally, if $\phi_{1r}, \phi_{1j} \in Y_3$ then $E_i \subset Y_3$ and is two-dimensional. The symmetry group Σ_3 of E_i is then generated by $S_3 = (S, \pi)$ and $R_3 = (R, \pi)$ and is again isomorphic to D_n. Hence the equivariant Hopf theorem implies a branch of solutions with symmetry $\Sigma_3 \subset D_n \times S^1$.

In all of these one-dimensional cases, the S_i, $i = 1, 2, 3$, symmetry implies that $c = 0$ so that the bifurcating branch of periodic solutions of (3.6) corresponds to periodic solutions of (3.1). However, any further bifurcations which break the reflectional symmetry will give rise to MTW solutions of (3.1).

In the case of the two-dimensional irreducible representations there are two eigenfunctions $\phi_1(x) = \phi_{1r}(x) \pm i\phi_{1j}(x)$ and $\phi_2(x) = \phi_{2r} \pm i\phi_{2j}(x)$, with $\phi_{1r}(x), \phi_{1j}(x), \phi_{2r}(x), \phi_{2j}(x) \in Y_{4m}$, corresponding to the double eigenvalues $\pm i$. Also $\phi_1(x) \in Y^s$ and $\phi_2(x) \in Y^a$ and so their structure is given in Theorem 2.2. The corresponding real eigenspace is $E_i = \mathrm{sp}\{\phi_{1r}(x), \phi_{1j}(x), \phi_{2r}(x), \phi_{2j}(x)\}$ which is four-dimensional. We identify the eigenspace E_i with $\mathbf{C}^2$ by

$$(x_1, y_1, x_2, y_2) \leftrightarrow x_1\phi_{1r}(x) + x_2\phi_{1j}(x) + y_1\phi_{2r}(x) + y_2\phi_{2j}(x), \tag{3.8}$$

where $z_j = x_j + iy_j$, $j = 1, 2$, and $(z_1, z_2) \in \mathbf{C}^2$. We then introduce the new coordinates $(\tilde{z}_1, \tilde{z}_2) = (\bar{z}_1 - i\bar{z}_2, z_1 - iz_2)$, so that in these coordinates θ acts diagonally on $\mathbf{C}^2$ [12]. The representation $\tilde{T}$ of $D_n \times S^1$ in these new coordinates is given by

$$\tilde{T}(S) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\tilde{T}(R) = \begin{pmatrix} e^{2\pi im/n} & 0 \\ 0 & e^{-2\pi im/n} \end{pmatrix},$$

$$\tilde{T}(\theta) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}.$$

Golubitsky, Stewart, and Schaeffer [12, Chap. XVIII] have shown that there are three isotropy subgroups of $D_n \times S^1$ acting on $\mathbf{C}^2$ which give two-dimensional fixed point spaces and these are given in Table I for odd n. Our example in the next section involves D_3 and so these are the appropriate spaces in this case.

In case (i), $\tilde{z}_1 = \tilde{z}_2$ and reverting to the original coordinates, this implies that $y_1 = y_2 = 0$. The identification (3.8) then implies that

$$(x_1, 0, x_2, 0) \leftrightarrow x_1\phi_{1r} + x_2\phi_{1j}.$$

TABLE I

Subgroups Giving Two-Dimensional Fixed Point Spaces for Hopf Bifurcation with $D_n \times S^1$ Symmetry for Odd n

Case	Isotropy subgroup	Fixed point space
(i)	$Z_2(S) = \{(I, 0), (S, 0)\}$	$\{(\tilde{z}_1, \tilde{z}_1)\}$
(ii)	$Z_2(S, \pi) = \{(I, 0), (S, \pi)\}$	$\{(\tilde{z}_1, -\tilde{z}_1)\}$
(iii)	$\tilde{Z}_n = \{(R, -2\pi/n)\}$	$\{(\tilde{z}_1, 0)\}$

Therefore the subgroup $\Sigma_1 = Z_2(S)$, reduces the four-dimensional space to a two-dimensional one and

$$Y^{\Sigma_1} \cap E_i = \text{sp}\{\phi_{1r}, \phi_{1j}\}. \tag{3.9}$$

Similarly, subgroups (ii) and (iii), respectively, imply that

$$Y^{\Sigma_2} \cap E_i = \text{sp}\{\phi_{2r}, \phi_{2j}\}, \tag{3.10}$$

$$Y^{\Sigma_3} \cap E_i = \text{sp}\{\phi_{1r} - \phi_{2j}, \phi_{1j} + \phi_{2r}\}, \tag{3.11}$$

where $\Sigma_2 = Z_2(S, \pi)$ and $\Sigma_3 = \tilde{Z}_n$. Hence, generically there are three branches of periodic solutions bifurcating from the primary branch. The solutions associated with the isotropy subgroup $Z_2(S)$ are contained in the symmetric subspace and have $c = 0$ while those associated with the subgroup $Z_2(S, \pi)$ are oscillating wave solutions, again with $c = 0$. However, solutions associated with the subgroup $\tilde{Z}_n$ do not have $c = 0$ and so the solutions are modulated travelling waves of the original Eq. (3.3). Therefore only two of the bifurcating branches give rise to periodic solutions. These results agree with Krupa [21], where he has shown that generically there are three branches bifurcating from the steady state, where two consist of periodic orbits and the third consists of two-tori.

The preceding analysis applies when m and n are coprime. If m and n are not coprime, i.e., $m = \tilde{m}l$, $n = \tilde{n}l$, then $(r_{2\pi/n})^{\tilde{n}} = I$ and $r_{2\pi/n}$ and s generate the group $D_{\tilde{n}}$, where $\tilde{n} = n/l$. Clearly $(r_{2\pi/n})^{\tilde{n}}$ acts trivially on E_i and the group generated by this element is Z_l, since $\tilde{n}l = n$. Thus, $E_i \subset Y^{Z_l}$ and, restricting to this fixed-point space, the group action is effectively that of $D_{\tilde{n}}$ and it is results for this group which must be considered. Such a bifurcation is essentially a rescaled version of a bifurcation occurring on a $D_{\tilde{n}}$ branch [4].

4. NUMERICAL RESULTS

To obtain a numerical solution of the Kuramoto–Sivashinsky equation, we use the spectral Galerkin method and thus we approximate $v(x, t)$ by

$$v_N(x, t) = \sum_{k=1}^{N} (a_k(t) \sin kx + b_k(t) \cos kx).$$

Note that there is no constant term as we assume that v has zero mean. Substituting this solution into (3.4) gives equations of the form

$$\dot{z} = f(z, c, \lambda), \tag{4.1}$$

where $z = [a_1, a_2, ..a_N, b_1, b_2, ..., b_N]^{\text{T}}$. We consider c as a parameter.

In practice, we use $\langle \ell, \bar{v}_N \rangle = 0$ as the phase condition for a suitable choice of ℓ. On steady state solutions v_N is independent of time. Thus $\bar{v}_N = v_N$ and the phase condition reduces to the simple algebraic equation $\langle \ell, v_N \rangle = 0$. On periodic solution branches, we implement it in AUTO as an integral constraint given by

$$\langle \ell, \bar{v} \rangle = \frac{1}{T} \int_0^T \langle l, v \rangle \, dt = 0.$$

The standard temporal phase condition used by AUTO is also included and the Eqs. (4.1), together with the two phase conditions are then solved for the time dependent functions $a_1, a_2, ..., a_N, b_1, b_2, ..., b_N$ and the scalar variables c and T, which are both considered to be parameters.

4.1. A Simple Hopf Bifurcation

We now consider a bifurcation for which Hyman, Nicolaenko, and Zaleski [13] obtained approximate numerical results by simply solving the initial value problem and describing the stable solutions which were found. In particular, they found a branch of stable steady state solutions which they referred to as strange fixed points (see Fig. 11 of [13]). These solutions have a reflectional symmetry and are 2π-periodic. Thus they have D_1 symmetry. Investigation of the steady state solutions shows that this is a branch of solutions which bifurcates from primary branch 2 at $\lambda = 52.89$.

The results of Hyman, Nicolaenko, and Zaleski [13] suggest that there are two bifurcations which occur. The first gives oscillating (periodic) solutions while the further bifurcation gives rise to what they call travelling beating waves which are similar except that they slowly drift. These solutions are shown in Fig. 12 and Fig. 13 of [13]. These results suggest that there may be a branch of oscillating waves (Fig. 12) and a branch of modulated travelling waves (Fig. 13). Hence, we applied our method to this problem taking $\ell = -\cos x$ and $N = 20$. A Hopf bifurcation which breaks the reflectional symmetry was found on the steady state branch at $\lambda = 83.7241$. This gives rise to a branch of oscillating waves with (S, π) symmetry. Following this branch, a secondary bifurcation was found at $\lambda = 85.672$. These results are in good agreement with those of Hyman, Nicolaenko, and Zaleski [13], who found "some kind of Hopf bifurcation" at $\lambda = 83.75$ and a further bifurcation at $\lambda = 86$. The branch of modulated travelling waves is stable up to a turning point at $\lambda = 90.81$ and then loses stability at this point. Hyman, Nicolaenko, and Zaleski [13] found a transition to chaotic behavior at $\lambda = 89$ which is quite near to this turning point. These solutions are shown in Fig. 1. Thus, we are able to give a precise description of the transitions observed in [13] in this region.
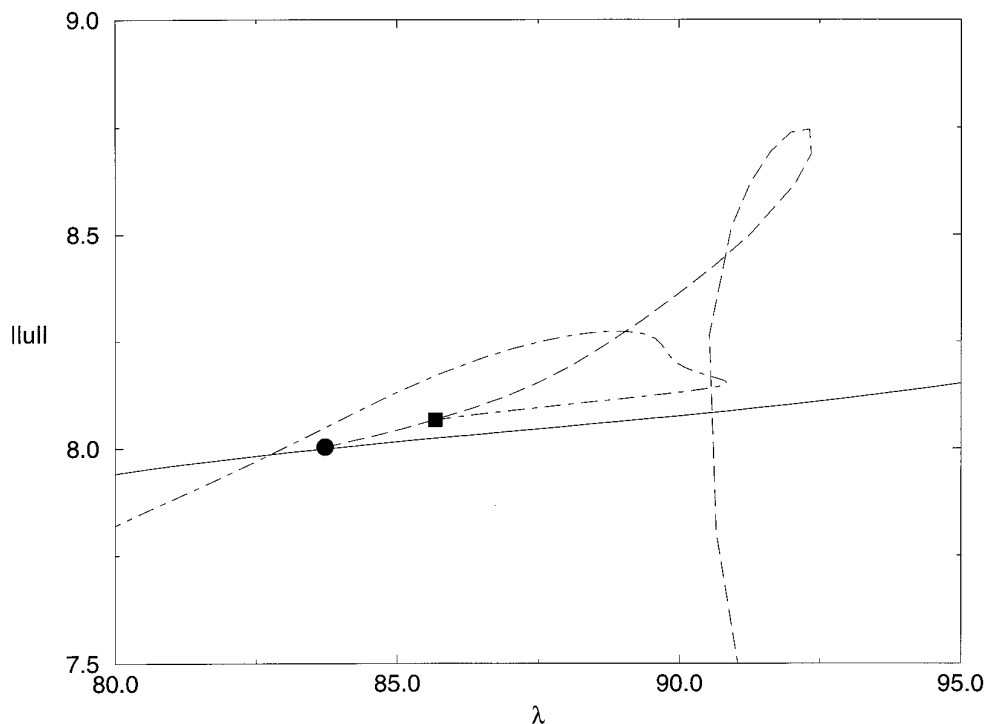
FIG. 1. Branches associated with the simple Hopf bifurcation: steady state solutions, ———; time periodic solutions, – – –; modulated travelling wave solutions, – – – – –; ●, bifurcation to oscillating waves; ■, bifurcation to modulated travelling waves.

The branch with D_1 symmetry is stable up to the bifurcation at $\lambda = 83.72$. Following the branch backwards in λ, a bifurcation to travelling waves is found at $\lambda = 70.03$, at which point the branch again loses stability. In this case the travelling wave branch is also unstable and so locally there are no stable solutions. This point was estimated by Hyman, Nicolaenko, and Zaleski [13] as $\lambda = 72$ and their results indicate that the solutions are chaotic for smaller values of λ.

4.2. A Multiple Hopf Bifurcation

There is a Hopf bifurcation on primary branch 3 at $\lambda = 66.751$ at which there are four eigenvalues on the imaginary axis with $\omega = 414.066$ and so this is clearly associated with the (only) two-dimensional irreducible representation of D_3. The numerical studies of Hyman, Nicolaenko, and Zaleski [13] indicate a bifurcation at $\lambda = 67.5$, but they mentioned that this point is not of Hopf type nor a classical homoclinic loop. Our numerical results give a clearer understanding of the solutions in this region.

Having detected the Hopf bifurcation, starting solutions for each of the three bifurcating branches of periodic solutions can be constructed using the eigenvectors in each of the two-dimensional fixed point spaces, as described in the previous section. More details can be found in [1].

All three bifurcating branches were computed numerically using $\ell = -\cos 3x$ and $N = 20$ and are shown, together with primary branch 3 in Fig. 2. As all the branches are supercritical, precisely one of them will be stable (see [12]). We show in the next section that the bifurcating branch 3 consisting of modulated travelling wave solutions is the stable branch. Note, however, that this branch soon loses stability at a turning point at $\lambda = 67.323$.

Hyman, Nicolaenko, and Zaleski [13] conjectured the existence of a bifurcation on the primary branch at $\lambda = 67.5$ but associated it with a perturbed Shilnikov homoclinic loop. However, our results show that the bifurcation on primary branch 3 occurs earlier at $\lambda = 66.751$. The stable bifurcating branch of modulated travelling waves then loses stability quite quickly at a turning point at $\lambda = 67.323$ and this is clearly the transition observed by Hyman, Nicolaenko, and Zaleski. Numerical results were obtained at $\lambda = 67.5$ in [13] which is just past the turning point. They considered the energy in different modes and in this norm, the quasiperiodic solutions on the bifurcating branch appear to be periodic. The period of the solution v at the turning point is $T = 0.017$ which is small due to the large value of ω at the multiple Hopf bifurcation point. Thus, in Figs. 4–7 of [13], the regions of very fast oscillations are where the trajectory approaches the region in which the quasiperiodic solutions existed before the turning point.
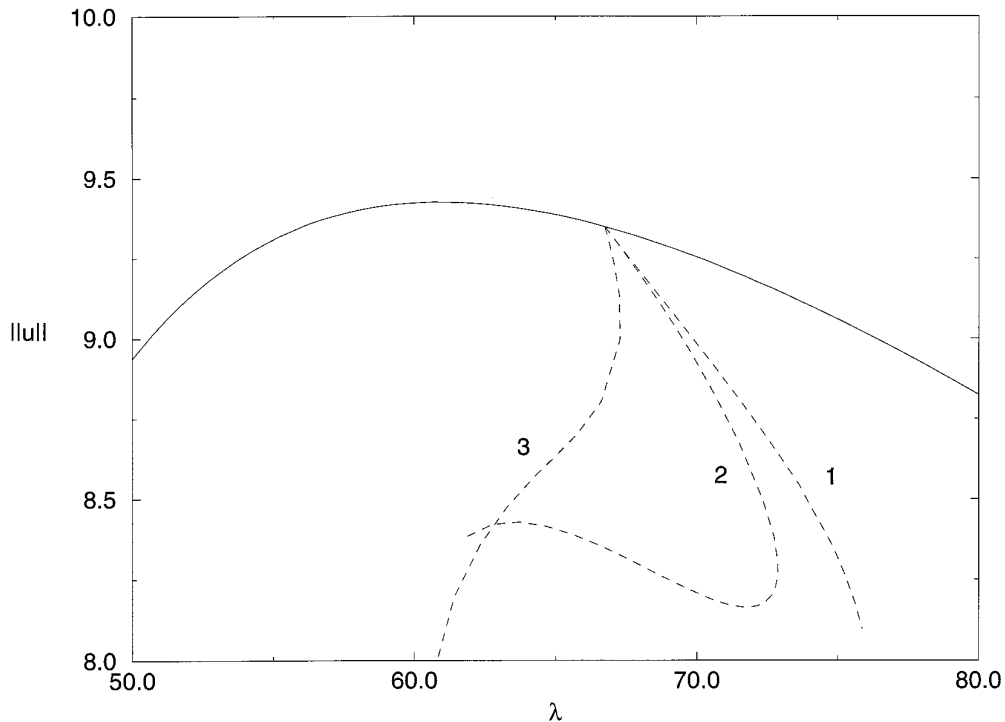
FIG. 2. Branches associated with the multiple Hopf bifurcation; branches 1 and 2 are time periodic solutions and branch 3 consists of modulated travelling wave solutions.

The solutions shown stay in this region for quite a large number of oscillations and then move away in a bursting type of behavior, before returning again in due course. The initial motion away from the shadow of the quasiperiodic solutions will be in a direction similar to the one-dimensional unstable manifold of the quasiperiodic solutions beyond the turning point. Note that in these regions of fast oscillation, modes 3 and 6 have quite large energies while the energy in mode 1 is quite low. The reason for this is that the Hopf bifurcation occurs from primary branch 3 on which the Fourier coefficients associated with modes 3 and 6 are nonzero, while those associated with mode 1 are zero. As the turning point on the branch of quasiperiodic solutions is quite close to the Hopf bifurcation point, then the energy in mode 1 has not had long to grow and is therefore still small. We note that the energy levels of the rapid oscillations are in agreement with our computed results for the quasiperiodic solutions at the turning point. We also note that these rapid oscillations are not associated with a spiral hyperbolic point as assumed in [13] but with the bifurcated quasiperiodic solutions.

Finally, part of the transient dynamics was considered at $\lambda = 68$ in [13] and a structure very close to a torus was found. This is clearly the shadow of the pair of two-tori which collided at the turning point.

4.3. Determination of Stability at the Multiple Hopf Bifurcation Point

A necessary condition for the existence of a stable bifurcating branch of solutions arising from the multiple Hopf bifurcation is that all three branches are supercritical [12]. Clearly from Fig. 2, this is the case here. In this situation, precisely one of the bifurcating branches is stable but there are conditions on cubic and quintic terms in the normal form equations to determine which is the stable branch. Thus, the computation of the branches is not sufficient to determine which is stable.

We consider the stability of the bifurcating branches in two ways. The first is an ad hoc method for estimating the important coefficients in the bifurcation equations while the second is a computational method for the accurate determination of these coefficients. Both methods give the same results, namely that the bifurcating branch of modulated travelling waves is the stable branch.

On primary branch 3, the Fourier coefficient a_1 is zero but on the bifurcating branches 1 and 3, a_1 is nonzero. We therefore assume that the branching equation associated with branch 1 has the form $c_1\mu a + c_2 a^3 = 0$ and for branch 3 is given by $c_1\mu a + c_3 a^3 = 0$, where $a = \|a_1(t)\|_\infty$, $\mu = \lambda - \lambda_0$, and $\lambda_0 = 66.751$ is the value of λ at the bifurcation

point. Comparing these equations and the branching equations in [12] gives

- $c_1 = \text{Re}(A_\mu(0)) < 0$,
- $c_2 = \text{Re}(2A_N(0) + B(0))$,
- $c_3 = \text{Re}(A_N(0) + B(0))$,

where $A_\mu(0)$, $A_N(0)$, and $B(0)$ are coefficients in the full, four-dimensional bifurcation equations. Our fom of the branching equations implies that on branch 1, $\mu = -(c_2/c_1)a^2 := \alpha_1 a^2$ and on branch 3, $\mu = -(c_3/c_1)a^2 := \alpha_2 a^2$. We estimate the coefficients α_1 and α_2 by assuming that there is a linear relationship between μ and a^2 near to the bifurcation point. By taking the bifurcation point ($\lambda = 66.751$, $a^2 = 0$) and one nearby point on the branches we can estimate the slope of this line, thus giving values of α_1 and α_2. For branch 1 we have the extra point

$$\lambda = 67.8226, \quad a^2 = 0.65936,$$

giving $\alpha_1 = 1.6248$. On branch 3 we have the extra point

$$\lambda = 67.2578, \quad a^2 = 1.1309,$$

giving $\alpha_2 = 0.44793$. Now $\text{Re}(B(0)) = 2c_3 - c_2 = -c_1(2\alpha_2 - \alpha_1)$ and so $\text{Re}(B(0)) < 0$. Since all the branches are supercritical, then $\text{Re}(A_N(0) + B(0)) > 0$. These are sufficient conditions for branch 3 to be the stable bifurcating branch. This method is not rigorous, since we have not derived the true bifurcation equations but it does give further backing to the assumption that branch 3 is the stable bifurcating branch.

A more formal approach to the determination of stability consists of accurately computing the coefficients of the normal form on the center manifold. After studying the stability of this reduced system it can be shown in many cases of practical interest that the bifurcation diagram and the stability of the bifurcating limit cycles are the same as for the normal form on the center manifold (see [12]).

We compute coefficients of the Taylor expansion using a numerical version of the Liapunov–Schmidt reduction. As the defining conditions used for accurate computation of Hopf bifurcation points are based on this reduction, computation of the higher order terms requires only small additional computational effort. The numerical reduction, in contrast to the classical Liapunov–Schmidt reduction, does not require exact knowledge of the center eigenspace (critical eigenvectors) and therefore it turns out to be numerically stable. We do not give all the details here but refer to [14, 15] for a more rigorous analysis of the equivariant reduction technique and applications to Hopf bifurcation.

The main object of the reduction is a so-called extended (bordered) system (see also [3, 6, 10, 16]). For a general problem given by $F(u, \lambda, \omega) \equiv -\omega u_t + f(u, \lambda) = 0$, $F: U \times \mathbf{R}^2 \to V$, one possible representation, that has been used in our computations, is given by

$$F(u_0 + v, \lambda_0 + \lambda, \omega_0 + \omega) + \mathbf{M}g = F(u_0, \lambda_0, \omega_0)$$
$$\mathbf{L}v = x, \quad (4.2)$$

where $\{\mathbf{L}, \mathbf{M}\}$ are suitable reduction operators with $\mathbf{L} \in \mathscr{L}(U, \mathbf{R}^m)$ and $M \in \mathscr{L}(\mathbf{R}^m, V)$. The operators $\{\mathbf{L}, \mathbf{M}\}$ are chosen in such a way that the mapping defined by (4.2) is regular in a neighborhood of $(u_0, \lambda_0, \omega_0)$, even if this point is a singular point of the mapping F with $\dim \text{Ker } F_u(u_0, \lambda_0, \omega_0) = m > 0$. As the mapping is regular, there exist (locally) smooth mappings $g: \mathbf{R}^m \times \mathbf{R}^2 \to \mathbf{R}^m$ and $v: \mathbf{R}^m \times \mathbf{R}^2 \to U$ that solve the system (4.2). Differentiation of the system (4.2) gives us equations for higher order derivatives of g and v. This procedure is well-defined as a consequence of the implicit function theorem. In the theory of the reduction it can be shown that the reduced bifurcation equation corresponds to $g(x, \lambda, \omega) = 0$ in a neighborhood of the origin $(0, 0)$ and the degeneracy condition is transformed as $g_x(0, 0, 0) = 0$. When a symmetry is present an equivariant reduced bifurcation equation is to be constructed and this imposes restrictions on the choice of $\{\mathbf{L}, \mathbf{M}\}$, namely

$$\mathbf{L}T_\gamma^U = \tilde{T}_\gamma^{(i)}\mathbf{L}, \quad T_\gamma^V \mathbf{M} = \mathbf{M}\tilde{T}_\gamma^{(i)}, \quad (4.3)$$

where $T_\gamma^U, T_\gamma^V, T_\gamma^{(i)}$ are representations of the symmetry group on U, V, and $\mathbf{R}^m$, respectively. An obvious choice consists of projections onto the irreducible representations of the group.

For the Kuramoto–Sivashinsky equation, we define (after rescaling time)

$$F(v, \lambda, \omega) := \omega v_t + 4v_{xxxx} + \lambda(v_{xx} + vv_x)$$

and we analyze the Hopf bifurcation as a singular point of F on the space of periodic functions, thus taking $U = C_{2\pi}^1(S^1, H_{2\pi}^2)$ and $V = C_{2\pi}^0(S^1, H_{2\pi}^2)$. We define the operators $\{\mathbf{L}, \mathbf{M}\}$ on the spaces of periodic functions U and V by

$$\mathbf{L}u = \frac{1}{2\pi}\int_0^{2\pi} Lu(s,.)e^{is}\,ds, \quad \mathbf{M}z = ze^{is}M + \bar{z}e^{-is}\overline{M},$$

and the operators $\{\mathbf{L}, \mathbf{M}\}$ are represented in the Fourier basis of $H_{2\pi}^2$ by

$$L_k^1 v = \frac{1}{2\pi}\int_0^{2\pi} v(x)\phi_k^s(x)\,dx, \quad M_k^1 z_1 = \text{Re}z_1\phi_k^s(x),$$

$$L_k^2 v = \frac{1}{2\pi}\int_0^{2\pi} v(x)\phi_k^c(x)\,dx, \quad M_k^2 z_2 = \text{Re}z_2\phi_k^c(x),$$

where $\phi_k^s(x)$ and $\phi_k^c(x)$ belong to the symmetric and anti-symmetric components of Y_{4m} (see Section 3); i.e., ϕ_k^s correspond to *sin* and ϕ_k^c to *cos* functions in the basis of Y_{4m}. It is easy to check that the operations $\{\mathbf{L}, \mathbf{M}\}$ satisfy the conditions (4.3) for the action of the symmetry group $O(2) \times S^1$. The presence of an extra zero eigenvalue, due to the $O(2)$ symmetry, means that the formulae for the computation of higher order derivatives of the reduced mapping h from [15] cannot be applied directly. The inverse of the operator $f_u(u_0, \lambda_0)$ (the linearization of the spatial part of the mapping F at a point on a primary branch of steady states) does not exist and it has to be replaced by a generalized inverse. The generalized inverse is defined by another extended system given by

$$\begin{pmatrix} f_u & m \\ l & 0 \end{pmatrix},$$

where the operators l and m are chosen in such a way that the bordered operator is regular. We note that the extended system used in Section 2 gives an explicit construction of such a ''regularization.'' Another way of eliminating the degeneracy due to the group orbit is to restrict to the symmetric subspace Y^s, since the tangent vector to the group orbit is antisymmetric. The reduction to the symmetric space enables us to use the algorithm for detection and accurate computation of Hopf points and bifurcation points as in the nondegenerate case (see [15], Appendix) but the stability analysis must be performed in the whole space.

We shall focus on the stability analysis of solutions bifurcating from the Hopf point at $\lambda = 66.751$. The local analysis requires information on the whole space $Y_{4m} \subset Y = \text{span}$ [sin kx, cos kx] and the Taylor expansion of the reduced mapping $h: \mathbf{R}^2 \times \mathbf{R}^2 \to \mathbf{C}^2$. The condition $h_x = 0$ corresponds to the presence of eigenvalues $\pm i\omega_0$ in the spectrum of $f_u(u_0, \lambda_0)$ and so it may serve as a test function that detects the Hopf point and symmetry breaking to a given isotypic component (depending on the choice of $L_k^{(i)}, M_k^{(i)}$). The possible types of symmetry breaking were described in Section 3. The computed values are summarized in Table II, where we denote the coordinates on the space $\mathbf{R}^2$ by x and y; i.e., $L_k^1 v = x$ and $L_k^2 v = y$. From these results, it is easy now to find $B(0) = -1.6230 \times 10^6 + i0.0366 \times 10^6$. Thus, we have $\text{Re}(A_N(0) + B(0)) > 0$, $\text{Re}(2A_N(0) + B(0)) > 0$, $\text{Re}(A_\lambda(0)) < 0$, and $\text{Re}(B(0)) < 0$ and, comparing with the local analysis in [12, p. 382, Theorem 3.1], we conclude that the $\tilde{Z}_3$ symmetric branch of modulated travelling waves is the stable bifurcating branch. This is in agreement with the first ad hoc method and is consistent with the numerical results of Hyman, Nicolaenko, and Zaleski [13].

TABLE II

Computed Coefficients of the Bifurcation Equations

h_x		$0.23800 \times 10^{-04} + $ i 0.56520×10^{-5}
h_{xx}		0.0
$h_{x\lambda}$	A_λ	$-0.41082 \times 10^2 + $ i 0.53291×10^1
h_{xxx}	$2A_N + B$	$0.16320 \times 10^7 - $ i 0.10748×10^7
h_y		$-0.19985 \times 10^{-6} - $ i 0.14028×10^{-5}
h_{yy}		0.0
$h_{y\lambda}$	A_λ	$-0.41082 \times 10^2 + $ i 0.53291×10^1
h_{yyy}	$A_N + B$	$0.44621 \times 10^4 + $ i 0.12941×10^5

5. CONCLUSIONS

A numerical method for dealing with Hopf bifurcations from steady state solutions which have D_n symmetry in problems with $O(2)$ symmetry has been described. The method has been applied to the Kuramoto–Sivashinsky equation in two different parameter regimes of interest. The numerical results obtained in both cases give a clear picture of the solution structure in these regions which had previously been explored using numerical simulation by Hyman, Nicolaenko, and Zaleski [13]. We note that in the first case of the simple Hopf bifurcation, the existence of a secondary bifurcation on the branch of oscillating waves near to the bifurcation from the steady state branch indicates that, if a second parameter were varied, then there is likely to be a steady state/Hopf mode interaction nearby. This has been shown to be the case by Amdjadi and Aston [2] who considered numerical methods for dealing with such mode interactions.

In the case of the multiple Hopf bifurcation, the stability of the bifurcating branches was determined by using an ad hoc method, based on the computed branches, and a more exact method, based on computing coefficients in the normal form. Since both methods gave the same results, this indicates that the ad hoc method may be useful in other situations where it is not possible to compute the normal form coefficients.

APPENDIX

We have transformed Eq. (3.1) to the form (3.3) due to numerical difficulties experienced with (3.1). We now attempt to explain the reasons for these problems.

If we substitute $U(x, t) = r_{\alpha(t)} u(x, t)$ into (3.1), we obtain

$$u_t + 4u_{xxxx} + \lambda(u_{xx} + \tfrac{1}{2} u_x^2) + cu_x = 0,$$

where $c(t) = \dot{\alpha}(t)$. There is not a unique solution for this equation, since if (u, c, λ) is a solution then $(u + \beta, c, \lambda)$

is also a solution. This is easily dealt with by requiring that u has zero mean, giving a modified equation similar to (3.2). However, there is also another family of solutions. Let $\tilde{u}_x = u_x + \beta$, which implies that $\tilde{u} = u + \beta x$, and so $u = \tilde{u} - \beta x$. Substituting in the above equation implies that

$$\tilde{u}_t + 4\tilde{u}_{xxxx} + \lambda(\tilde{u}_{xx} + \tfrac{1}{2}(\tilde{u}_x)^2) \\ + (c - 2\beta)\tilde{u}_x + \beta^2 - c\beta = 0.$$

Now the spectral Galerkin method used effectively ignores the constant component of the equation, assuming that it is always zero and so $(u + \beta x, c - 2\beta, \lambda)$ is also a solution of the numerical equations. Of course, $u + \beta x$ is not periodic, but it does have a Fourier series representation involving *sine* functions. This one-parameter family of solutions then causes the Jacobian matrix at a steady state solution to be (almost) singular, which is the cause of the numerical problems.

The reflectional symmetry for this equation is $su(x) = u(-x)$ which implies that symmetric functions are even. In practice the numerical method works well when u is an even function, which is to be expected since the one-parameter family of solutions involves the odd function x which is therefore excluded. However, the method does not converge to a simple steady state solution which is not even.

By defining $v = u_x$, the first family of solutions does not arise and the second family is given by $v + \beta$. Requiring v to have zero mean then eliminates this degeneracy.

REFERENCES

1. F. Amdjadi, Ph.D thesis, University of Surrey, 1994 (unpublished).
2. F. Amdjadi and P. J. Aston, Numerical methods for steady state/ Hopf mode interactions, *Int. J. Bif. Chaos*, to appear.
3. P. Ashwin, K. Böhmer, and Z. Mei, *Math. Comput.* **64,** 649 (1994).
4. P. J. Aston, Scaling laws and bifurcation, in *Singularity Theory and Its Applications, Warwick 1989, Part II*, edited by M. Roberts and I. Stewart, Lecture Notes in Mathematics, Vol. 1463 (Springer-Verlag, Heidelberg, 1991), p. 1.
5. P. J. Aston, *SIAM J. Math. Anal.* **22,** 181 (1991).
6. P. J. Aston, A. Spence, and W. Wu, *SIAM J. Appl. Math.* **52,** 792 (1992).
7. P. J. Aston, A. Spence, and W. Wu, An investigation of the bifurcation from travelling waves to modulated travelling waves, in *Bifurcation and Symmetry*, edited by E. Allgower, K. Böhmer, and M. Golubitsky (Birkhäuser, Basel, 1992), p. 35.
8. G. W. Bluman and S. Kumei, *Symmetries and Differential Equations* (Springer-Verlag, New York, 1989).
9. M. Dellnitz, *IMA J. Numer. Anal* **12,** 429 (1991).
10. M. Dellnitz and B. Werner, *J. Comput. Appl. Math.* **26,** 97 (1989).
11. E. J. Doedel, *Congr. Numer.* **30,** 265 (1981).
12. M. Golubitsky, I. Stewart, and D. G. Schaeffer *Singularities and Groups in Bifurcation Theory, Vol. II,* Appl. Math. Sci., Vol. 69 (Springer-Verlag, New York, 1988).
13. J. M. Hyman, B. Nicolaenko, and S. Zaleski, *Physica D* **23,** 265 (1986).
14. V. Janovský and P. Plecháč, Numerical applications of equivariant reduction techniques, in *Bifurcation and Symmetry,* edited by E. Allgower, K. Böhmer, and M. Golubitsky (Birkhäuser, Basel, 1992), p. 203.
15. V. Janovský and P. Plecháč, Local numerical analysis of Hopf bifurcation, *SIAM J. Num. Anal.* **33,** 1150 (1996).
16. A. D. Jepson and A. Spence, *SIAM J. Math. Anal.* **20,** 39 (1989).
17. I. G. Kevrekidis, B. Nicolaenko, and J. C. Scovel, *SIAM J. Appl. Math.* **50,** 760 (1989).
18. Y. Kuramoto, *Prog. Theor. Phys. Suppl.* **64,** 346 (1978).
19. Y. Kuramoto, *Chemical Oscillations, Waves and Turbulence* (Springer-Verlag, New York, 1984).
20. Y. Kuramoto, *Prog. Theor. Phys. Suppl.* **71,** 1182 (1984).
21. M. Krupa, *SIAM J. Math. Anal.* **21,** 1453 (1990).
22. A. S. Landsberg and E. Knobloch, *Phys. Lett. A* **159,** 17 (1991).
23. A. S. Landsberg and E. Knobloch, *Phys. Lett A* **179,** 316 (1993).
24. M. R. E. Proctor and N. O. Weiss, *Geophys. Astrophys. Fluid Dyn.* **70,** 137 (1992).
25. P. C. Matthews, M. R. E. Proctor, A. M. Rucklidge, and N. O. Weiss, *Phys. Lett. A* **183,** 69 (1993).
26. T. Shlang and G. I. Sivashinksy, *J. Phys.* **43,** 459 (1982).
27. G. I. Sivashinsky, *Acta Astronaut.* **4,** 1177 (1977).
28. G. I. Sivashinsky, *Ann. Rev. Fluid Mech.* **15,** 179 (1983).
29. G. I. Sivashinsky and D. M. Michelson, *Prog. Theor. Phys.* **63,** 2112 (1980).
30. B. Werner and A. Spence, *SIAM J. Numer. Anal.* **21,** 388 (1984).